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Kemmer Algebras and Rotation Groups

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Abstract

Some previous results indicating a connection between the Kemmer formulation of meson theory and the group of rotations in six dimensions are generalised. A correspondence between the irreducible representations of the Kemmer algebra K_N and the skewsymmetric tensor representations of rotations in $N+1$ dimensions is established.

1. *Introduction*

Let γ_μ ($\mu = 1, \ldots 5$) be the five Dirac matrices, satisfying

$$\gamma_{(\mu}\gamma_{\nu)} = \delta_{\mu\nu} \tag{1.1}$$

Define $\gamma_6 = -i$ and use capital Latin indices for the six-fold indices, thus

$$\gamma_A = (\gamma_\mu, -i)$$

We define

$$\bar{\gamma}_A = (\gamma_\mu, i)$$

which is analogous to the concept of 'quaternion conjugation' in the Pauli algebra. Define the sets of matrices

$$1, \gamma_A, \gamma_{AB} = \gamma_{[A}\bar{\gamma}_{B]}, \gamma_{ABC} = \gamma_{[AB}\gamma_{C]}, \ldots, \gamma_{ABCDEF} \tag{1.2}$$

The matrices (1.2) are not linearly independent. The sets occur in 'mutually dual' pairs:

$$\begin{aligned} \gamma_{123456} &= i \\ \gamma_{12345} &= i\gamma_6 \\ &\ldots \\ \gamma_{12} &= i\gamma_{6543} \\ \gamma_1 &= i\gamma_{65432} \\ 1 &= i\gamma_{654321} \end{aligned} \tag{1.3}$$

and we also have all the relations that follow from (1.3) by permutation of the indices 12 … 6 with a minus sign introduced for odd permutations. Thus

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each matrix of the set $\gamma_{AB\ldots}$ with n suffixes is equal to i times one with $6-n$ suffixes. The set γ_{ABC} is self-dual:

$$i\gamma_{123} = \gamma_{456} \qquad \text{(and permutations)}$$

so that it contains $\frac{1}{2}\binom{6}{3} = 10$ linearly independent matrices.

Further, any matrix with an *even* number of suffixes is contained in one of the two adjacent *odd* sets, and vice versa: e.g.

$$\left.\begin{aligned} \gamma_{\mu\nu} &= i\gamma_{\mu\nu 6} \\ \gamma_{\mu 6} &= i\gamma_{\mu} \end{aligned}\right\} (\mu, \nu = 1, \ldots 5)$$

expresses γ_{AB} in terms of γ_A and γ_{ABC}.

Thus two alternative linearly independent sets occurring in (1.2) are

(A) γ_A, γ_{ABC} $(6 + 10 = 16 \text{ matrices})$

(B) $1, \gamma_{AB}$ $(1 + 15 = 16 \text{ matrices})$

The matrices $\gamma_{AB}/2$ and $\bar{\gamma}_{AB}/2$ (where $\bar{\gamma}_{AB} = \bar{\gamma}_{[A}\gamma_{B]}$) are easily shown to be generators of two inequivalent irreducible representations S and $\bar{S}$ of rotations in six dimensions. The coefficients of a 4×4 matrix A expressed as a linear combination of the set (A) transform like a vector and a rank 3 self-dual skewsymmetric tensor under the transformation law $A \to SA\bar{S}^{-1}$ and the coefficients of a matrix B expressed in terms of (B) transform as a scalar and a rank two tensor under $B \to SB\bar{S}^{-1}$.

These ideas are easily generalisable to $2^\nu \times 2^\nu$ matrices (rank 2 spinors) in $2\nu + 2 = M$ dimensions; we can associate with a skewsymmetric tensor of rank k, $\varphi_{A_1 \ldots A_k}$, a rank two spinor

$$\Phi = \varphi_{A_1 \ldots A_k}\gamma_{A_1 \ldots A_k} \qquad (A_1, A_2 \ldots = 1, \ldots M) \tag{1.4}$$

We discussed only the case $\nu = 2$ for the sake of simplicity. A more general treatment is given elsewhere (Lord, 1972).

2. *Generalised Kemmer Equations*

Let Φ be a rank two spinor in M-dimensional space, of the form (1.4) (without loss of generality we can take $k \leqslant M/2$) and let p_A $(A = 1, \ldots M)$ be a vector. Consider the covariant equation

$$p_A \bar{\gamma}_A \Phi = 0 \tag{2.1}$$

From

$$\left.\begin{aligned} \gamma_{(A}\bar{\gamma}_{B)} &= \delta_{AB} \\ \bar{\gamma}_{(A}\gamma_{B)} &= \delta_{AB} \end{aligned}\right\} \tag{2.2}$$

we can obtain

$$\begin{aligned} \bar{\gamma}_A \gamma_B &= \bar{\gamma}_{AB} + \delta_{AB} \\ \bar{\gamma}_A \gamma_{BC} &= \bar{\gamma}_{ABC} + 2\delta_{A[B}\bar{\gamma}_{C]} \\ \bar{\gamma}_A \gamma_{BCD} &= \bar{\gamma}_{ABCD} + 3\delta_{A[B}\bar{\gamma}_{CD]} \\ &\ldots \end{aligned} \tag{2.3}$$

and also

$$\begin{aligned} \gamma_B \bar{\gamma}_A &= \gamma_{BA} + \delta_{BA} \\ \gamma_{BC} \bar{\gamma}_A &= \gamma_{BCA} + 2\gamma_{[B}\delta_{C]A} \\ \gamma_{BCD} \bar{\gamma}_A &= \gamma_{BCDA} + 3\gamma_{[BC}\delta_{D]A} \end{aligned} \tag{2.4}$$

It therefore follows that, if Φ corresponds to a tensor with an *even* number of suffixes, so that its transformation law is $S\Phi S^{-1}$, the two equations

$$p_A \bar{\gamma}_A \Phi = 0 \quad \text{and} \quad \Phi p_A \gamma_A = 0 \tag{2.5}$$

are equivalent to each other† and to the tensor equations

$$\begin{aligned} p_{[A}\, \varphi_{BC\ldots]} &= 0 \\ p_A\, \varphi_{ABC\ldots} &= 0 \end{aligned} \tag{2.6}$$

Of course, if the rank of the tensor is $M/2$ the tensor will be self-dual and the two equations (2.6) will be the same equation.

If Φ corresponds to a tensor with an odd number of indices, its transformation law will be $S\Phi \bar{S}^{-1}$ and the two equations

$$p_A \bar{\gamma}_A \Phi = 0 \quad \text{and} \quad \Phi p_A \bar{\gamma}_A = 0 \tag{2.7}$$

will be equivalent to each other† and to tensor equations of the form (2.6).

Now equations (2.5) can be written (by multiplication with γ_B),

$$\begin{aligned} p_A \gamma_{BA} \Phi + p_B \Phi &= 0 \\ \Phi p_A \gamma_{AB} + p_B \Phi &= 0 \end{aligned} \tag{2.8}$$

Adding,

$$p_A G_{BA} \Phi + p_B \Phi = 0 \tag{2.9}$$

where G_{AB} are the infinitesimal generators for the representation Φ of the rotation group. That is, the G_{AB} are infinitesimal generators of a completely skewsymmetric tensor representation. Setting $p_M = m$ and using Greek indices for the range $1 \ldots M-1$ and remembering that $\gamma_M = -i$ (2.9) becomes

$$p_\mu \beta_\mu \Phi + im\Phi = 0 \tag{2.10}$$

where

$$\beta_\mu = iG_{M\mu} \tag{2.11}$$

$$\beta_\mu \Phi = \tfrac{1}{2}(\gamma_\mu \Phi - \Phi\gamma_\mu) = \tfrac{1}{2}(\gamma_\mu \otimes 1 - 1 \otimes \gamma_\mu{}^T)\Phi \tag{2.12}$$

Since $\frac{1}{2}(\gamma_\mu \otimes 1 - 1 \otimes \gamma_\mu{}^T) = B_\mu$ satisfies the defining relations of the Kemmer algebra K_{M-1}

$$B_\mu B_\nu B_\rho + B_\rho B_\nu B_\mu = \delta_{\mu\nu} B_\rho + \delta_{\rho\nu} B_\mu \tag{2.13}$$

† An exception occurs when the tensor has rank $M/2 - 1$, in which case the first equation (2.6) can be split into two equations (its self-dual and antiself-dual parts). The pair of equations (2.5) (or (2.7)) are then distinct, and must both be postulated to obtain the whole of (2.6).

it follows that (2.10) is a Kemmer equation. The irreducible set of matrices β_μ generate an irreducible representation of the Kemmer algebra K_{M-1}.

The equations (2.7) can be treated similarly. The only change in the argument is that γ_{AB} is replaced by $\bar{\gamma}_{AB}$ in the second equation (2.8).

Thus we have established a (1 – 1) correspondence between the irreducible representations of the Kemmer algebra K_{M-1} and the irreducible representations of the rotation group in M dimensions given by completely skewsymmetric tensors. If β_μ are the matrices of an irreducible representation of K_N, then $i\beta_\mu$ and $[\beta_\mu, \beta_\nu]$ are generators of a skewsymmetric tensor representation of rotations in $N+1$ dimensions.

TABLE 1

				1		(1, 1)
			1		3	
		1		4		(3, 3)
	1		5		10	
1		6		15		(10, 10)
				...		

The Nth row in Table 1 gives the dimensionalities of irreducible representations of K_N (Kemmer, 1943). The bracketed pairs are the 'associated' pairs of inequivalent representations with the same dimensionality. We can also regard the Nth row as a list of the dimensionalities of skewsymmetric tensors in $(N+1)$-space. The associated pairs of representations of the Kemmer algebra correspond to the decomposition of a skewsymmetric tensor into self-dual and antiself-dual parts when its rank is half the dimensionality of the space.

We have strictly speaking demonstrated the result only for even M. However, for the odd-dimensional spaces (even-dimensional Kemmer algebras) the result is easily extended by considering the decomposition of a rank k tensor under projection and the similar decomposition of representations of Kemmer algebras—each number in Table 1 decomposes to the two entries immediately above it. This generalises the study of equation (2.1) previously given for the case $M = 6$ (Lord, 1971).

Note that putting $p_M = m$, which reduced (2.9) to (2.10) gives the equations in the following form when applied to (2.6):

$$\left.\begin{aligned} km\varphi_{\mu\nu\rho\ldots} &= p_{[\mu}\chi_{\nu\rho\ldots]} \\ m\chi_{\nu\rho\ldots} &= -p_\mu \varphi_{\mu\nu\rho\ldots} \end{aligned}\right\} \tag{2.14}$$

$$\left.\begin{aligned} p_{[\mu}\varphi_{\nu\rho\ldots]} &= 0 \\ p_\mu \chi_{\mu\nu\rho\ldots} &= 0 \end{aligned}\right\} \tag{2.15}$$

where k is the rank of the tensor, the Greek indices are $(M-1)$-fold and $\chi_{\mu\nu\rho\ldots} = \varphi_{M\mu\nu\rho\ldots}$. Equations (2.15) are just simple consequences of (2.14). Equations (2.14) are the ones given by Kemmer. When $2k = M$ the $(M-1)$-

dimensional tensor $\varphi_{\mu\nu\rho\ldots}$ is the dual of $\chi_{\mu\nu\ldots}$, or *minus* the dual of $\chi_{\mu\nu\ldots}$, corresponding to whether $\varphi_{ABC\ldots}$ is self-dual or antiself-dual, and corresponding to the two associated representations of K_{M-1}.

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